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Weighted Moduli of Smoothness and Spline Spaces

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In this paper we study relations between moduli of smoothness with the step-weight function φ and the best approximation by splines with knots uniformly distributed according to the measure with density $1/\varphi(x)$. The direct and converse results are obtained for a class of step-weight functions, containing $\varphi(x)=\sqrt{x(1-x)}$; it is well known that the modulus of smoothness corresponding to this φ is related to the best polynomial approximation. As a consequence, we obtain relations between the best polynomial and spline approximations.

1. INTRODUCTION

The aim of this paper is to establish a relationship between the moduli of smoothness with variable step function and approximation by spline spaces with suitable knots. We consider the class of moduli of smoothness on $I=[0,1]$ corresponding to step-weight functions $\varphi(x)\sim x^{\beta(0)}(1-x)^{\beta(1)}$. This class contains the important step-weight function $\varphi(x)=\sqrt{x(1-x)}$. The modulus of smoothness with this particular φ appears naturally in the characterization of the best polynomial approximation in $L^p(I)$ (see [8, Chapter 7]). It appears as well in the characterization of the order of approximation by Bernstein, Kantorovich, and Durrmeyer operators (which are positive polynomial operators).

In this paper we relate the modulus of smoothness of order m with the step-weight function φ to the order of approximation by splines of degree m (i.e., of order $m+1$) with the simple knots uniformly distributed according to the measure with density $1/\varphi(x)$. The direct and converse results are obtained (Theorems 4.3 and 4.4); moreover, it is shown that the orthogonal projections onto the spline spaces under consideration give the best order of approximation in all the $L^p(I)$ norms. Analogous results are proved for some local positive spline operators.

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It should be noted that for $\varphi(x) \equiv 1$ (corresponding to $\beta(0) = \beta(1) = 0$) we obtain classical moduli of smoothness in L^p norm, and their relation with the order of approximation by spline functions with dyadic knots was studied earlier by Ciesielski (see for example [2]).

The case of $\varphi(x) = \sqrt{x(1-x)}$ is discussed in detail in the last section. It is well known that the corresponding knots are extreme points of Tchebyshev polynomials of the first kind. The results of Theorems 4.3 and 4.4, combined with the direct and converse results for the best polynomial approximation, give Marchaud type inequalities between the best polynomial and spline approximations (see Corollary 5.1). Consequently, we get the same order of best approximation by polynomials and appropriate splines for the generalized Hölder classes. We should mention that spline spaces with knots close to the extreme points of the Tchebyshev polynomials mentioned above appear in [5] in the proof of the equivalence of the K-functional and the modulus of smoothness corresponding to φ.

The paper is organized as follows. In Section 2 we recall the definition and main properties of the moduli of smoothness with variable step function. In Section 3 we describe the spline spaces and operators under consideration. Section 4 contains the results for general $\varphi(x) \sim x^{\beta(0)}(1-x)^{\beta(1)}$, and in Section 5 the case of $\varphi(x) = \sqrt{x(1-x)}$ is discussed in detail.

To shorten the notation, the following abbreviations are used. For $a, b \in R$, we write $a \vee b = \max(a, b)$, $a \wedge b = \min(a, b)$ and $a \sim b$ if there are two constants $c_1, c_2 > 0$ such that $c_1 a \leqslant b \leqslant c_2 a$. The Lebesgue measure of the set A is denoted by $|A|$. Moreover, by C we denote a constant, the value of which may vary from line to line.

2. WEIGHTED MODULI OF SMOOTHNESS

Denote $I = [0, 1]$; for $1 \leqslant p < \infty$, $L^p(I)$ is the space of real-valued functions defined on I, integrable with pth power, with the usual norm $\|f\|_p = (\int_0^1 |f(x)|^p \, dx)^{1/p}$; by $C(I)$ we denote the space of continuous functions on I, with the usual supremum norm.

Let us recall the concept of weighted moduli of smoothness (for more details see [8]). For $f: I \to R$, $m \in N$ and $h \in R$ denote by $\bar{\Delta}_h^m f(x)$ the symmetric difference of f of order m with the step h, i.e.,

$$\bar{\Delta}_h^m f(x) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} f(x + mh/2 - ih),$$

with the convention that $\bar{\Delta}_h^m f(x) = 0$ if $x \pm mh/2 \notin I$.

We are interested in the moduli of smoothness with the step of the difference depending on the point. Let $\varphi: I \to [0, \infty)$; the function φ is called *an admissible step-weight function* if it satisfies the following conditions.

I. φ is measurable and $\varphi \sim 1$ locally, i.e., for any proper subinterval $[a, b] \subset (0, 1)$ there is a constant C such that $1/C \leqslant \varphi(x) \leqslant C$ for all $x \in [a, b]$.

II. There are numbers $\beta(0), \beta(1) \geqslant 0$ such that

$$\varphi(x) \sim x^{\beta(0)} \quad \text{as} \quad x \to 0 \qquad \text{and} \qquad \varphi(x) \sim (1-x)^{\beta(1)} \quad \text{as} \quad x \to 1.$$

III. There are C_0 and h_0 such that for $0 < h \leqslant h_0$ and every finite interval $E \subset I$

$$|\{x: x \pm h\varphi(x) \in E, x \in I\}| \leqslant C_0 \, |E|.$$

Condition III guarantees the continuity of the modulus of smoothness as a functional over $L^p(I)$; if $\beta(0) < 0$ or $\beta(1) < 0$ in II, then condition III is not satisfied. For the detailed discussion of conditions I–III, see [8].

Let φ be an admissible step-weight function. Then for $f: I \to R$ the modulus of smoothness of f of order m and with the step-weight φ in the $L^p(I)$ norm is defined as

$$\omega_{\varphi, p}^{(m)}(f, t) = \sup_{0 < h \leqslant t} \|\bar{\Delta}_{h\varphi}^m f\|_p.$$

Now, we list the properties of the modulus of smoothness $\omega_{\varphi, p}^{(m)}(f, t)$, which are needed later on. The first of these properties is the equivalence of the modulus of smoothness $\omega_{\varphi, p}^{(m)}(f, t)$ and the appropriate K-functional. The K-functional under consideration is

$$K_{\varphi, p}^{(m)}(f, t) = \inf\{\|f - g\|_p + t^m \|\varphi^m \cdot g^{(m)}\|_p : g \in W_{p, \varphi}^m(I)\},$$

with

$$W_{p, \varphi}^m(I) = \{g \in L^p(I) \cap AC_{loc}^{m-1}(I): \|\varphi^m \cdot g^{(m)}\|_p < \infty\},$$

where $AC_{loc}^{m-1}(I)$ is the space of functions with $g^{(m-1)}$ absolutely continuous on each subinterval of I, and for $p = \infty$, $L^\infty(I)$ is replaced by $C(I)$. According to [8, Theorem 2.1.1], we have

THEOREM 2.1. *Let φ be an admissible step-weight function and let $m \in N$, $1 \leqslant p \leqslant \infty$ be given. Then there are C and t_0 such that for $f \in L^p(I)$ ($f \in C(I)$ in case $p = \infty$)*

$$\frac{1}{C}\,\omega^{(m)}_{\varphi,p}(f,t) \leqslant K^{(m)}_{\varphi,p}(f,t) \leqslant C\omega^{(m)}_{\varphi,p}(f,t) \qquad \text{for} \quad 0 < t \leqslant t_0.$$

Let $1 \leqslant p \leqslant \infty$, $m \in N$ be given. The following properties of $\omega^{(m)}_{\varphi,p}(f,t)$ are used frequently without further reference; their proofs can be found in Chapter 4 of [8].

(2.1) Suppose that φ_1, φ_2 are two admissible step-weight functions, satisfying $\varphi_1(x) \leqslant C\varphi_2(x)$ for $x \in I$. Then there are M and t_0 such that

$$\omega^{(m)}_{\varphi_1,p}(f,t) \leqslant M\omega^{(m)}_{\varphi_2,p}(f,t) \qquad \text{for} \quad 0 \leqslant t \leqslant t_0.$$

In particular, if $\varphi_1(x) \sim \varphi_2(x)$, then $\omega^{(m)}_{\varphi_1,p}(f,t) \sim \omega^{(m)}_{\varphi_2,p}(f,t)$ for $0 \leqslant t \leqslant t_0$.

(2.2) There are C and t_0 such that

$$\omega^{(m)}_{\varphi,p}(f,\lambda t) \leqslant C\lambda^m \omega^{(m)}_{\varphi,p}(f,t) \qquad \text{for} \quad 0 \leqslant t \leqslant \lambda t \leqslant t_0.$$

(2.3) There are C and t_0 such that for $f \in W^m_{p,\varphi}(I)$

$$\omega^{(m)}_{\varphi,p}(f,t) \leqslant Ct^m \, \|\varphi^m \cdot f^{(m)}\|_p \qquad \text{for} \quad 0 \leqslant t \leqslant t_0.$$

(2.4) There are C and t_0 such that

$$\omega^{(m+1)}_{\varphi,p}(f,t) \leqslant C\omega^{(m)}_{\varphi,p}(f,t) \qquad \text{for} \quad 0 \leqslant t \leqslant t_0.$$

(2.5) Marchaud type inequality: There are C and t_0 such that

$$\omega^{(m)}_{\varphi,p}(f,t) \leqslant Ct^m \left(\|f\|_p + \int_t^{t_0} \frac{\omega^{(m+1)}_{\varphi,p}(f,u)}{u^{m+1}}\,du \right) \qquad \text{for} \quad 0 \leqslant t \leqslant t_0.$$

In particular, it follows from (2.4) and (2.5) that for $0 < \alpha < m$ the conditions $\omega^{(m)}_{\varphi,p}(f,t) = O(t^\alpha)$ and $\omega^{(m+1)}_{\varphi,p}(f,t) = O(t^\alpha)$ are equivalent.

3. SPLINE SPACES ASSOCIATED WITH $\varphi(x)$

3.1. *Knot Sequences and Their Properties*

Let φ be an admissible step-weight function and let $\pi_{n,\varphi} = \{t_{n,k}, k \in \Upsilon_{n,\varphi}\}$, for $n \in N$, be the sequence of knots uniformly distributed in I with step $1/n$ according to the measure with density $1/\varphi(x)$. More precisely, the set of indices $\Upsilon_{n,\varphi}$ is specified below, and for $k \in \Upsilon_{n,\varphi}$, the point $t_{n,k}$ is defined as the solution of an appropriate equation.

Case I. $0 \leqslant \beta(0), \beta(1) < 1$. Then

$$\Upsilon_{n,\varphi} = \{0, 1, ..., n\}, \qquad \int_0^{t_{n,k}} \frac{dx}{\varphi(x)} = \frac{k}{n} \int_0^1 \frac{dx}{\varphi(x)}.$$

Case II. $0 \leqslant \beta(0) < 1, \beta(1) \geqslant 1$. Then

$$\Upsilon_{n,\varphi} = \{0, 1, 2, ...\}, \qquad \int_0^{t_{n,k}} \frac{dx}{\varphi(x)} = \frac{k}{n}.$$

Case III. $\beta(0) \geqslant 1, 0 \leqslant \beta(1) < 1$. Then

$$\Upsilon_{n,\varphi} = \{..., -2, -1, 0\}, \qquad \int_{t_{n,k}}^1 \frac{dx}{\varphi(x)} = \frac{|k|}{n}.$$

Case IV. $\beta(0), \beta(1) \geqslant 1$. Then

$$\Upsilon_{n,\varphi} = Z, \qquad \begin{cases} t_{n,0} = \dfrac{1}{2}, \\ \displaystyle\int_{t_{n,k}}^{1/2} \frac{dx}{\varphi(x)} = \frac{|k|}{n} & \text{for} \quad k < 0, \\ \displaystyle\int_{1/2}^{t_{n,k}} \frac{dx}{\varphi(x)} = \frac{k}{n} & \text{for} \quad k > 0. \end{cases}$$

Moreover, let

$$\Upsilon^*_{n,\varphi} = \{k \in \Upsilon_{n,\varphi} : k-1 \in \Upsilon_{n,\varphi}\}$$
$$\Upsilon^\circ_{n,\varphi} = \{k \in \Upsilon^*_{n,\varphi} : k-1, k+1 \in \Upsilon^*_{n,\varphi}\}.$$

For $k \in \Upsilon^*_{n,\varphi}$ define

$$I_{n,k} = (t_{n,k-1}, t_{n,k}), \qquad \lambda_{n,k} = |I_{n,k}| = t_{n,k} - t_{n,k-1}.$$

The condition $k \in \Upsilon^\circ_{n,\varphi}$ means that the interval $I_{n,k}$ does not touch the boundary of I, and there is no singularity of φ at the endpoints of $I_{n,k}$.

In the sequel, we consider orthogonal projections onto the spaces of spline functions of order $m+1$ with the knots $\pi_{n,\varphi}$ and we need the fact that the L^p-norms of these projections are uniformly bounded in n. Now, we present some estimates for the ratios $\lambda_{n,k}/\lambda_{n,l}$ for the partitions $\pi_{n,\varphi}$ (see Proposition 3.1 below), which imply the requested bound for the norms of the projections under consideration.

PROPOSITION 3.1. *Let φ be an admissible step-weight function on I, and let $\pi_{n,\varphi}$ be the associated knot sequence. Then there is a constant C, depending only on φ, such that*

$$\frac{\varphi(x)}{n} \leqslant C\lambda_{n,k} \qquad \text{for} \quad x \in I_{n,k}, \quad k \in \Upsilon^*_{n,\varphi}, \tag{3.1}$$

$$\lambda_{n,k} \leqslant C\frac{\varphi(x)}{n} \qquad \text{for} \quad x \in I_{n,k}, \quad k \in \Upsilon^\circ_{n,\varphi}. \tag{3.2}$$

Moreover, we have the following estimates for the ratio $\lambda_{n,k}/\lambda_{n,l}$.

1. *If $\beta(0) \neq 1 \neq \beta(1)$, then there are C and γ such that*

$$\frac{1}{C}\cdot\frac{1}{(1+|k-l|)^\gamma} \leqslant \frac{\lambda_{n,k}}{\lambda_{n,l}} \leqslant C(1+|k-l|)^\gamma \qquad \text{for} \quad k, l \in \Upsilon^*_{n,\varphi}, \quad n \in N.$$

2. *If $\beta(0) = 1 = \beta(1)$, then there are C and $q_n \geqslant 1$ with $\lim_{n\to\infty} q_n = 1$ such that*

$$\frac{1}{C}\cdot q_n^{-|k-l|} \leqslant \frac{\lambda_{n,k}}{\lambda_{n,l}} \leqslant Cq_n^{|k-l|} \qquad \text{for} \quad k, l \in \Upsilon^*_{n,\varphi}, \quad n \in N.$$

3. *If $\beta(0) = 1$ and $\beta(1) \neq 1$, or $\beta(0) \neq 1$ and $\beta(1) = 1$, then there are C, γ and $q_n \geqslant 1$ with $\lim_{n\to\infty} q_n = 1$ such that for all $n \in N$ and $k, l \in \Upsilon^*_{n,\varphi}$*

$$\frac{1}{C}\,\frac{1}{q_n^{|k-l|}(1+|k-l|)^\gamma} \leqslant \frac{\lambda_{n,k}}{\lambda_{n,l}} \leqslant Cq_n^{|k-l|}(1+|k-l|)^\gamma.$$

Proof. Let us consider the case $0 \leqslant \beta(0),\ \beta(1) < 1$. At first, let $0 \leqslant k \leqslant (2/3)\,n$. As $\varphi(x) \sim x^{\beta(0)}(1-x)^{\beta(1)}$, we have for these k's

$$t_{n,k} \sim \left(\frac{k}{n}\right)^{1/(1-\beta(0))}, \tag{3.3}$$

which gives $\lambda_{n,1} = t_{n,1} - t_{n,0} \sim (1/n)^{1/(1-\beta(0))}$. Denote

$$F(u) = \int_0^u \frac{1}{\varphi(t)}\,dt, \qquad a = \int_0^1 \frac{1}{\varphi(t)}\,dt, \qquad G(u) = F^{-1}(u).$$

Clearly, $G'(u) = \varphi(G(u))$, so for $k > 1$ and $((k-1)/n)\,a \leqslant u \leqslant (k/n)\,a$ we have $G'(u) \sim (k/n)^{\beta(0)/(1-\beta(0))}$, and by the mean value theorem

$$\lambda_{n,k} = t_{n,k} - t_{n,k-1} = G\left(\frac{k}{n}a\right) - G\left(\frac{k-1}{n}a\right) \sim \frac{1}{n}\cdot\left(\frac{k}{n}\right)^{\beta(0)/(1-\beta(0))}. \tag{3.4}$$

Similarly, we check that for $(1/3)\, n \leqslant k \leqslant n$

$$1 - t_{n,k} \sim \left(\frac{n-k}{n}\right)^{\beta(1)/(1-\beta(1))} \qquad \text{and} \qquad \lambda_{n,k} \sim \frac{1}{n} \cdot \left(\frac{n-k+1}{n}\right)^{\beta(1)/(1-\beta(1))}. \tag{3.5}$$

The required bounds for the ratio $\lambda_{n,k}/\lambda_{n,l}$, as well as inequalities (3.1) and (3.2), follow from (3.3)–(3.5).

The other cases are treated analogously, so the details are omitted. ∎

3.2. *Spline Spaces and Projections*

Let φ be an admissible step-weight function on I; for $n, m \in N$ put

$$\Upsilon^{(m)}_{n,\varphi} = \{k \in \Upsilon_{n,\varphi} : k+1 \in \Upsilon_{n,\varphi}\} \cup \{-m, ..., -1\}.$$

Let $\{N^{(m,\varphi)}_{n,i}, i \in \Upsilon^{(m)}_{n,\varphi}\}$ be the sequence of *B*-splines with the knots $\pi_{n,\varphi}$ of order $m+1$, normalized in such a way that they form a partition of unity, i.e.,

$$N^{(m,\varphi)}_{n,i}(t) = (t_{n,i+m+1} - t_{n,i})[t_{n,i}, ..., t_{n,m+i+1}; (\cdot - t)^m_+] \qquad \text{for} \quad i \in \Upsilon^{(m)}_{n,\varphi},$$

where $[s_0, ..., s_l; f]$ denotes the divided difference of order l of f, taken at the points $s_0, ..., s_l$, and the points $t_{n,j}$ for $j \notin \Upsilon_{n,\varphi}$ are given by the following rule: if $j \notin \Upsilon_{n,\varphi}$ and $j<0$ (which can happen only if $\beta(0)<1$), then $t_{n,j}=0$, while if $j \notin \Upsilon_{n,\varphi}$ and $j>0$ (which can occure only if $\beta(1)<1$), then $t_{n,j}=1$.

Let us mention some of the properties of the functions $N^{(m,\varphi)}_{n,i}$ (for details see for example [1] or [11]).

(3.6) $N^{(m,\varphi)}_{n,i}(t) \geqslant 0$, supp $N^{(m,\varphi)}_{n,i} = [t_{n,i}, t_{n,i+m+1}]$.

(3.7) On any subinterval $I' \subset I$, the functions $N^{(m,\varphi)}_{n,i}$ which are non-trivial on I', are linearly independent over this interval.

(3.8) $\sum_{i \in \Upsilon^{(m)}_{n,\varphi}} N^{(m,\varphi)}_{n,i}(t) = 1$ for all $t \in (0, 1)$.

(3.9) Let $\xi_{n,i} = \|N^{(m,\varphi)}_{n,i}\|_1 = (t_{n,i+m+1} - t_{n,i})/(m+1)$; then there is a constant $C>0$ such that for any sequence $\{a_i, i \in \Upsilon^{(m)}_{n,\varphi}\}$ and $1 \leqslant p \leqslant \infty$

$$C\left(\sum_{i \in \Upsilon^{(m)}_{n,\varphi}} \xi_{n,i}\, |a_i|^p\right)^{1/p} \leqslant \left\| \sum_{i \in \Upsilon^{(m)}_{n,\varphi}} a_i N^{(m,\varphi)}_{n,i}(t) \right\|_p \leqslant \left(\sum_{i \in \Upsilon^{(m)}_{n,\varphi}} \xi_{n,i}\, |a_i|^p\right)^{1/p}.$$

(3.10) For $i \in \Upsilon_{n,\varphi}^{(m)}$ let $M_{n,i}^{(m,\varphi)} = N_{n,i}^{(m,\varphi)} / \| N_{n,i}^{(m,\varphi)} \|_1$; then

$$\frac{d}{dt} N_{n,i}^{(m,\varphi)} = \begin{cases} -M_{n,i+1}^{(m-1,\varphi)} & \text{if} \quad t_{n,i} = \cdots = t_{n,i+m} = 0, \\ M_{n,i}^{(m-1,\varphi)} & \text{if} \quad t_{n,i+1} = \cdots = t_{n,i+m+1} = 1, \\ M_{n,i}^{(m-1,\varphi)} - M_{n,i+1}^{(m-1,\varphi)} & \text{otherwise.} \end{cases}$$

Let

$$\mathcal{S}_n^{(m,\varphi)} = \left\{ f \colon f = \sum_{i \in \Upsilon_{n,\varphi}^{(m)}} a_i N_{n,i}^{(m,\varphi)},\ a_i \in R \right\}.$$

Moreover, let $\mathcal{S}_{n,p}^{(m,\varphi)} = \mathcal{S}_n^{(m,\varphi)} \cap L^p(I)$ (note that if $\beta(0), \beta(1) < 1$ then $\dim \mathcal{S}_n^{(m,\varphi)} = n+m$ and $\mathcal{S}_{n,p}^{(m,\varphi)} = \mathcal{S}_n^{(m,\varphi)}$ for all p; if either $\beta(0) \geqslant 1$ or $\beta(1) \geqslant 1$, then the space $\mathcal{S}_n^{(m,\varphi)}$ is of infinite dimension). Let $P_n^{(m,\varphi)}$ be the orthogonal projection onto $\mathcal{S}_{n,2}^{(m,\varphi)}$, and let $\mathcal{P}_n^{(m,\varphi)}(x,y)$ be the Dirichlet kernel of $P_n^{(m,\varphi)}$. It follows from the exponential estimates for $\mathcal{P}_n^{(m,\varphi)}(x,y)$ (see Theorem 3.2 below) that the formula,

$$P_n^{(m,\varphi)} f(x) = \int_0^1 \mathcal{P}_n^{(m,\varphi)}(x,y)\, f(y)\, dy, \tag{3.11}$$

defines a bounded linear operator on $L^p(I)$ for all $1 \leqslant p \leqslant \infty$; moreover, if g is a polynomial of degree $\leqslant m$, then $P_n^{(m,\varphi)} g = g$.

Let $\mathbf{H} = [h_{i,j}, i, j \in \Upsilon_{n,\varphi}^{(m)}]$ be the inverse to the Gram matrix $\mathbf{G} = [(N_{n,i}^{(m,\varphi)}, N_{n,j}^{(m,\varphi)}), i, j \in \Upsilon_{n,\varphi}^{(m)}]$. Then

$$\mathcal{P}_n^{(m,\varphi)}(x,y) = \sum_{i,j \in \Upsilon_{n,\varphi}^{(m)}} h_{i,j} N_{n,i}^{(m,\varphi)}(x)\, N_{n,j}^{(m,\varphi)}(y). \tag{3.12}$$

It was proved by de Boor (cf. [1]) that there are $0 < \vartheta < 1$ and $C > 0$, depending on m only, such that

$$|h_{i,j}| \leqslant C \frac{\vartheta^{|i-j|}}{\sqrt{\xi_{n,i} \cdot \xi_{n,j}}}. \tag{3.13}$$

(It should be noted that the special case of this result, for splines with dyadic knots, was obtained earlier by Domsta in [9].)

This estimate and Proposition 3.1 imply the following theorem.

THEOREM 3.2. *Let φ be an admissible step-weight function and $m \in N$. Then there are $n_0 \in N$, C and $0 < \theta < 1$ such that*

$$|\mathcal{P}_n^{(m,\varphi)}(x,y)| \leqslant C \frac{\theta^{|k-l|}}{\lambda_{n,k} \vee \lambda_{n,l}} \qquad \textit{for} \quad n \geqslant n_0, \quad x \in I_{n,k}, \quad y \in I_{n,l}. \tag{3.14}$$

This estimate implies that there is a finite constant M such that for all p, $1 \leqslant p \leqslant \infty$, and $n \geqslant n_0$

$$\|P_n^{(m,\varphi)}\|_p = \|P_n^{(m,\varphi)} : L^p(I) \to \mathscr{S}_{n,p}^{(m,\varphi)}\| \leqslant M. \tag{3.15}$$

Proof. The bound for the ratio $\lambda_{n,k}/\lambda_{n,l}$ from Proposition 3.1, (3.6), and (3.9) implies that if $I_{n,k} \subset \operatorname{supp} N_{n,i}^{(m,\varphi)}$, then $\xi_{n,i} \sim \lambda_{n,k}$. Inequality (3.14) follows from the above mentioned bound for $\lambda_{n,k}/\lambda_{n,l}$, (3.6), (3.12), and (3.13); clearly, it is sufficient to take any θ such that $\vartheta < \theta < 1$. It follows from (3.14) that there is a constant M, depending on φ, such that for all $n \geqslant n_0$ and $t \in I$

$$\int_0^1 |\mathscr{P}_n^{(m,\varphi)}(x,t)|\,dx \leqslant M \qquad \text{and} \qquad \int_0^1 |\mathscr{P}_n^{(m,\varphi)}(t,y)|\,dy \leqslant M,$$

which by the standard argument implies (3.15). ∎

Other Spline Operators. Together with the projections $P_n^{(m,\varphi)}$, we consider local positive spline operators and piecewise linear interpolating operator.

For $m, n \in N$ put

$$\mathscr{L}_n^{(m,\varphi)}(x,y) = \sum_{i \in \Upsilon_{n,\varphi}^{(m)}} N_{n,i}^{(m,\varphi)}(x)\, M_{n,i}^{(m,\varphi)}(y) \tag{3.16}$$

and

$$L_n^{(m,\varphi)} f(x) = \int_0^1 \mathscr{L}_n^{(m,\varphi)}(x,y)\, f(y)\,dy. \tag{3.17}$$

$L_n^{(m,\varphi)} f$ is well defined for $f \in L^p(I)$, $1 \leqslant p \leqslant \infty$, and if f is a nonnegative function, then $L_n^{(m,\varphi)} f(x)$ is nonnegative as well. Moreover, as B-splines $\{N_{n,i}^{(m,\varphi)}, i \in \Upsilon_{n,\varphi}^{(m)}\}$ form a partition of unity, we have $L_n^{(m,\varphi)} 1 = 1$ and $\int_0^1 f(x)\,dx = \int_0^1 L_n^{(m,\varphi)} f(x)\,dx$. Thus $L_n^{(m,\varphi)}$ takes probability densities supported on I into probability densities, and this property makes these operators useful for nonparametric density estimation (cf. [4]).

PROPOSITION 3.3. *Let φ be an admissible step-weight function and $m \in N$. For $n \in N$ and $s, t \in I$, let $i, j \in \Upsilon_{n,\varphi}^*$ be such that $s \in I_{n,i}$ and $t \in I_{n,j}$, and define*

$$l_n^{(m,\varphi)}(s,t) = \begin{cases} \dfrac{1}{\lambda_{n,i}} & \text{if} \quad |i-j| \leqslant m, \\ 0 & \text{if} \quad |i-j| > m. \end{cases}$$

Then there are a constant C *and* $n_0 \in N$ *such that for* $n \geqslant n_0$ *and* $s, t \in I$

$$\mathscr{L}_n^{(m,\varphi)}(s,t) \leqslant C l_n^{(m,\varphi)}(s,t). \tag{3.18}$$

Moreover, for all $1 \leqslant p \leqslant \infty$

$$\|L_n^{(m,\varphi)}\|_p = \|L_n^{(m,\varphi)} : L^p(I) \to \mathscr{S}_{n,p}^{(m,\varphi)}\| = 1. \tag{3.19}$$

Proof. Inequality (3.18) follows from formula (3.16) by arguments analogous to the proof of inequality (3.14). Since $\mathscr{L}_n^{(m,\varphi)}$ is symmetric, nonnegative, and $L_n^{(m,\varphi)} 1 = 1$, we get (3.19). ∎

Finally, let for $n \in N$ and $f \in C(I)$, $U_n^{(\varphi)} f$ be the piecewise linear function, interpolating f at the knots $\pi_{n,\varphi}$, i.e.

$$U_n^{(\varphi)} f \in \mathscr{S}_n^{(1,\varphi)}, \qquad U_n^{(\varphi)} f(t_{n,k}) = f(t_{n,k}) \qquad \text{for} \quad k \in \Upsilon_{n,\varphi}.$$

We are interested in the bounds for the orders of approximation by the operators $P_n^{(m,\varphi)}$, $L_n^{(m,\varphi)}$ and $U_n^{(\varphi)}$ in the terms of moduli of smoothness with the step-weight function φ. Note the differences between the operators $P_n^{(m,\varphi)}$, $L_n^{(m,\varphi)}$ and $U_n^{(\varphi)}$: $P_n^{(m,\varphi)}$ reproduces polynomials of degree m, while $L_n^{(m,\varphi)}$ reproduces only constant functions. Therefore, we can obtain the bounds for the order of approximation by $P_n^{(m,\varphi)}$ in terms of moduli of smoothness of order $m+1$, and for the order of approximation by $L_n^{(m,\varphi)}$, in general, we can get only modulus of smoothness of order 1; however, for the particular step-weight function $\varphi(x) = \sqrt{x(1-x)}$ we are able to obtain the bound for the order of approximation by $L_n^{(m,\varphi)}$ in terms of modulus of smoothness of order 2 (cf. Theorem 5.2). On the other hand, $U_n^{(\varphi)}$ is well defined for continuous functions, but it reproduces linear functions and we prove the bound for the order of approximation by $U_n^{(\varphi)}$ in terms of modulus of smoothness of order 2.

4. WEIGHTED MODULI OF SMOOTHNESS AND ORDERS OF APPROXIMATION

4.1. *Order of Approximation by* $P_n^{(m,\varphi)}$

LEMMA 4.1 (Jackson-Type Inequality for $P_n^{(m,\varphi)}$). *Let* φ *be an admissible step-weight function and* $m \in N$. *Then there are finite constant* C *and* $n_0 \in N$ *such that for all* $n \geqslant n_0$, $1 \leqslant p \leqslant \infty$ *and all* $f \in W_{p,\varphi}^{m+1}(I)$

$$\|f - P_n^{(m,\varphi)} f\|_p \leqslant \frac{C}{n^{m+1}} \|\varphi^{m+1} \cdot f^{(m+1)}\|_p. \tag{4.1}$$

Proof. Note that it is enough to prove inequality (4.1) for $f \in C^{m+1}(I)$ and $1 \leqslant p < \infty$, with the constant C independent of p.

Let us start with the case $0 \leqslant \beta(0), \beta(1) < 1$; to simplify the notation, put $\beta(0) = \beta_0$, $\beta(1) = \beta_1$. In this case $\Upsilon^*_{n,\varphi} = \{1, ..., n\}$, and it follows from inequalities (3.1) and (3.2) from Proposition 3.1 that

$$\frac{1}{n}\varphi(x) \sim \lambda_{n,k} \qquad \text{for} \quad 2 \leqslant k \leqslant n-1, \qquad x \in I_{n,k}. \tag{4.2}$$

Moreover, we have

$$t_{n,1} = \lambda_{n,1} \sim \left(\frac{1}{n}\right)^{1/(1-\beta_0)}, \qquad 1 - t_{n,n-1} = \lambda_{n,n} \sim \left(\frac{1}{n}\right)^{1/(1-\beta_1)} \tag{4.3}$$

(cf. formulae (3.3)–(3.5) in the proof of Proposition 3.1).

For $f \in C^{m+1}(I)$ we get from Taylor's formula

$$f(x) = g(f, x) + \int_0^1 f^{(m+1)}(y)\, W_m(x, y)\, dy,$$

where $W_m(x, y) = \operatorname{sgn}(x-y) \cdot (x-y)^m/(2m!)$ and $g(f, \cdot)$ is a polynomial of degree m. As the operator $P_n^{(m,\varphi)}$ reproduces polynomials of degree $\leqslant m$, we have

$$P_n^{(m,\varphi)} f(x) = g(f, x) + \int_0^1 f^{(m+1)}(y)\, P_n^{(m,\varphi)}(W_m(\cdot, y))(x)\, dy,$$

and consequently

$$f(x) - P_n^{(m,\varphi)} f(x) = \int_0^1 f^{(m+1)}(y)(W_m(x, y) - P_n^{(m,\varphi)}(W_m(\cdot, y))(x))\, dy. \tag{4.4}$$

To calculate $P_n^{(m,\varphi)}(W_m(\cdot, y))(x)$ note that for given y the function $W_m(\cdot, y)$ is a polynomial of degree m on the intervals $[0, y]$ and $[y, 1]$. Since $P_n^{(m,\varphi)}$ reproduces polynomials of degree $\leqslant m$, we obtain by (3.11)

$$\begin{aligned} P_n^{(m,\varphi)}(W_m(\cdot, y))(x) &= \frac{1}{2m!}(x-y)^m + 2\int_0^y \mathscr{P}_n^{(m,\varphi)}(x, t)\, W_m(t, y)\, dt \\ &= \frac{-1}{2m!}(x-y)^m + 2\int_y^1 \mathscr{P}_n^{(m,\varphi)}(x, t)\, W_m(t, y)\, dt. \end{aligned}$$

Using the first of these representations of $P_n^{(m,\varphi)}(W_m(\cdot, y))(x)$ for $y \leqslant x$ and the second one for $y \geqslant x$, we get from (4.4)

$$f - P_n^{(m,\varphi)} f = \frac{1}{m!}\,(R_n^{(m,\varphi)} f - T_n^{(m,\varphi)} f), \tag{4.5}$$

with

$$R_n^{(m,\varphi)} f(x) = \int_0^x \int_0^y f^{(m+1)}(y)\, \mathscr{P}_n^{(m,\varphi)}(x, t)(t-y)^m\, dt\, dy,$$

$$T_n^{(m,\varphi)} f(x) = \int_x^1 \int_y^1 f^{(m+1)}(y)\, \mathscr{P}_n^{(m,\varphi)}(x, t)(t-y)^m\, dt\, dy.$$

We present the proof of the bound for $\|T_n^{(m,\varphi)} f\|_p$; the appropriate bound for $\|R_n^{(m,\varphi)} f\|_p$ can be obtained in a similar way. Splitting the integral defining $T_n^{(m,\varphi)} f(x)$ (i.e., the integral over $[x, 1]$) into sum of integrals over $[x, 1] \cap I_{n,n}$, $[x, 1] \cap [t_{n,1}, t_{n,n-1}]$ and $[x, 1] \cap I_{n,1}$, and using again (for $y \in [x, 1] \cap I_{n,1}$) the fact that $P_n^{(m,\varphi)}$ reproduces polynomials of degree $\leqslant m$, we obtain the following decomposition of $T_n^{(m,\varphi)} f$ as

$$T_n^{(m,\varphi)} f = T_{n,0}^{(m,\varphi)} f + T_{n,1}^{(m,\varphi)} f + T_{n,2}^{(m,\varphi)} f + T_{n,3}^{(m,\varphi)} f,$$

where

$$T_{n,0}^{(m,\varphi)} f(x) = \int_{x \vee t_{n,n-1}}^1 \int_y^1 f^{(m+1)}(y)\, \mathscr{P}_n^{(m,\varphi)}(x, t)(t-y)^m\, dt\, dy,$$

$$T_{n,1}^{(m,\varphi)} f(x) = \begin{cases} \displaystyle\int_{x \vee t_{n,1}}^{t_{n,n-1}} \int_y^1 f^{(m+1)}(y)\, \mathscr{P}_n^{(m,\varphi)}(x, t)(t-y)^m\, dt\, dy \\ \qquad \text{for} \quad 0 \leqslant x \leqslant t_{n,n-1}, \\ 0 \qquad \text{for} \quad t_{n,n-1} < x \leqslant 1, \end{cases}$$

$$T_{n,2}^{(m,\varphi)} f(x) = \begin{cases} \displaystyle -\int_x^{t_{n,1}} \int_0^y f^{(m+1)}(y)\, \mathscr{P}_n^{(m,\varphi)}(x, t)(t-y)^m\, dt\, dy \\ \qquad \text{for} \quad 0 \leqslant x \leqslant t_{n,1}, \\ 0 \qquad \text{for} \quad t_{n,1} < x \leqslant 1, \end{cases}$$

$$T_{n,3}^{(m,\varphi)} f(x) = \begin{cases} \displaystyle\int_x^{t_{n,1}} f^{(m+1)}(y)(x-y)^m\, dy & \text{for} \quad 0 \leqslant x \leqslant t_{n,1}, \\ 0 & \text{for} \quad t_{n,1} < x \leqslant 1. \end{cases}$$

Each of the terms $\|T_{n,i}^{(m,\varphi)} f\|_p$, $i = 0, 1, 2, 3$, is treated separately.

Let us start with the estimate of $\|T_{n,3}^{(m,\varphi)} f\|_p$. Denote $a = \int_0^1 du/\varphi(u)$; by the definition of $t_{n,1}$ we have $\int_0^{t_{n,1}} du/\varphi(u) = a/n$. Thus, using the integral Jensen's inequality (with the measure $(n/a)(dy/\varphi(y))$ on $[x, t_{n,1}]$) and (4.3) we get

$$\begin{aligned}\|T_{n,3}^{(m,\varphi)}f\|_p^p &\leqslant \int_0^{t_{n,1}} \left(\int_x^{t_{n,1}} |f^{(m+1)}(y)|\, \varphi(y)(y-x)^m \frac{dy}{\varphi(y)} \right)^p dx \\ &\leqslant \frac{a^{p-1}}{n^{p-1}} \int_0^{t_{n,1}} \int_x^{t_{n,1}} |f^{(m+1)}(y)|^p\, \varphi(y)^{p-1}\, (y-x)^{pm}\, dy\, dx \\ &\leqslant \frac{a^{p-1}}{n^{p-1}} \int_0^{t_{n,1}} |f^{(m+1)}(y)|^p\, \varphi(y)^{p-1}\, y^{pm+1}\, dy \\ &\leqslant \frac{a^{p-1}}{n^{p-1}} \lambda_{n,1}^{(1-\beta_0)(pm+1)} \int_0^{t_{n,1}} |f^{(m+1)}(y)|^p \\ &\quad \times \varphi(y)^{p-1}\, y^{\beta_0(pm+1)}\, dy \\ &\leqslant \frac{C^p}{n^{p(m+1)}} \int_0^{t_{n,1}} |f^{(m+1)}(y)|^p\, \varphi(y)^{p(m+1)}\, dy.\end{aligned}$$

Now, we estimate $\|T_{n,2}^{(m,\varphi)}f\|_p$. Inequality (3.14) of Theorem 3.2 implies that

$$|\mathscr{P}_n^{(m,\varphi)}(x,t)| \leqslant \frac{C}{\lambda_{n,1}} \qquad \text{for} \quad x, t \in I_{n,1}.$$

Using this estimate, Jensen's inequality (again with the measure $(n/a)(dy/\varphi(y))$ on $[x, t_{n,1}]$) and (4.3) we obtain

$$\begin{aligned}\|T_{n,2}^{(m,\varphi)}f\|_p^p &\leqslant \frac{C^p}{\lambda_{n,1}^p} \frac{a^{p-1}}{n^{p-1}} \int_0^{t_{n,1}} \int_x^{t_{n,1}} \left(\int_0^y (y-t)^m\, dt \right)^p \\ &\quad \times |f^{(m+1)}(y)|^p\, \varphi(y)^{p-1}\, dy\, dx \\ &\leqslant \frac{C^p}{n^{p-1}} \lambda_{n,1}^{(1-\beta_0)(pm+1)} \int_0^{t_{n,1}} y^{\beta_0(pm+1)} \varphi(y)^{p-1} \\ &\quad \times |f^{(m+1)}(y)|^p\, dy \\ &\leqslant \frac{C^p}{n^{p(m+1)}} \int_0^{t_{n,1}} \varphi(y)^{p(m+1)}\, |f^{(m+1)}(y)|^p\, dy.\end{aligned}$$

Let us estimate $\|T_{n,0}^{(m,\varphi)}f\|_p$. Applying Jensen's inequality (with the measure $(n/a)(dy/\varphi(y))$ on $[t_{n,n-1}, 1]$), inequality (3.14) from Theorem 3.2 and (4.3) we get

$$\begin{aligned}\|T_{n,0}^{(m,\varphi)}f\|_p^p &\leqslant \frac{a^{p-1}}{n^{p-1}} \int_0^1 \int_{x \vee t_{n,n-1}}^1 \left(\int_y^1 |\mathscr{P}_n^{(m,\varphi)}(x,t)|\, (t-y)^m\, dt \right)^p \\ &\quad \times |f^{(m+1)}(y)|^p\, \varphi(y)^{p-1}\, dy\, dx\end{aligned}$$

$$\leqslant \frac{C^p}{n^{p-1}} \sum_{l=1}^{n} \frac{\theta^{p(n-l)}}{(\lambda_{n,l} \vee \lambda_{n,n})^p} \int_{I_{n,l}} \int_{t_{n,n-1}}^{1} (1-y)^{p(m+1)} \times |f^{(m+1)}(y)|^p \varphi(y)^{p-1} \, dy \, dx$$

$$\leqslant \frac{C^p}{n^{p-1}} \lambda_{n,n}^{(1-\beta_1)(pm+1)} \int_{t_{n,n-1}}^{1} (1-y)^{\beta_1(pm+1)} \varphi(y)^{p-1} \times |f^{(m+1)}(y)|^p \, dy$$

$$\leqslant \frac{C^p}{n^{p(m+1)}} \int_{t_{n,n-1}}^{1} \varphi(y)^{p(m+1)} |f^{(m+1)}(y)|^p \, dy.$$

It remains to estimate $\| T_{n,1}^{(m,\varphi)} f \|_p$. Introduce

$$k_n(x, t) = \theta^{|k-l|/2}, \qquad p_n(x, t) = \frac{\theta^{|k-l|/2}}{\lambda_{n,k} \vee \lambda_{n,l}} \qquad \text{for} \quad x \in I_{n,k}, \quad t \in I_{n,l}, \tag{4.6}$$

with θ from Theorem 3.2. Then we can rewrite (3.14) as

$$|\mathscr{P}_n^{(m,\varphi)}(x, t)| \leqslant C p_n(x, t) \, k_n(x, t). \tag{4.7}$$

Moreover, let for $0 \leqslant x \leqslant t_{n,n-1}$

$$z(x) = \int_{x \vee t_{n,1}}^{t_{n,n-1}} \int_{y}^{1} (t-y)^m \, k_n(x, t) \, dt \, dy.$$

Applying (4.7) and then the integral Jensen's inequality (with the measure $(t-y)^m k_n(x, t) \, dt \, dy / z(x)$ on the set $\{(t, y): x \vee t_{n,1} \leqslant y \leqslant t_{n,n-1}, y \leqslant t \leqslant 1\}$) we obtain

$$\| T_{n,1}^{(m,\varphi)} f \|_p^p = \int_0^{t_{n,n-1}} \left| \int_{x \vee t_{n,1}}^{t_{n,n-1}} \int_y^1 f^{(m+1)}(y) \times \mathscr{P}_n^{(m,\varphi)}(x, t)(t-y)^m \, dt \, dy \right|^p dx$$

$$\leqslant C^p \int_0^{t_{n,n-1}} z(x)^{p-1} \int_{x \vee t_{n,1}}^{t_{n,n-1}} \int_y^1 |f^{(m+1)}(y)|^p \times p_n(x, t)^p \, k_n(x, t)(t-y)^m \, dt \, dy \, dx$$

$$\leqslant C^p \int_{t_{n,1}}^{t_{n,n-1}} |f^{(m+1)}(y)|^p \times \left(\int_0^y \int_y^1 z(x)^{p-1} \, p_n(x, t)^p \, k_n(x, t)(t-y)^m \, dt \, dx \right) dy. \tag{4.8}$$

Let us start with the estimate of $z(x)$. Let $x \in I_{n,k}$. By the definition of $k_n(x,t)$ (cf. (4.6)) and the estimates for the ratio $\lambda_{n,i}/\lambda_{n,j}$ as given in Proposition 3.1 we get

$$\begin{aligned} z(x) &\leqslant \sum_{l=k}^{n-1} \sum_{j=l}^{n} \int_{t_{n,l-1}}^{t_{n,l}} \int_{t_{n,j-1}}^{t_{n,j}} |t-y|^m \, k_n(x,t)\, dt\, dy \\ &\leqslant \sum_{l=k}^{n-1} \sum_{j=l}^{n} \lambda_{n,l} \lambda_{n,j} (\lambda_{n,j} + \cdots + \lambda_{n,l})^m \, \theta^{(j-k)/2} \\ &\leqslant C \sum_{l=k}^{n-1} \lambda_{n,l}^{m+2} \theta^{(l-k)/2} \sum_{j=l}^{n} (1+(j-l))^{\gamma+m(\gamma+1)} \, \theta^{(j-l)/2} \\ &\leqslant C \lambda_{n,k}^{m+2} \sum_{l=k}^{n-1} (1+(l-k))^{\gamma(m+2)} \, \theta^{(l-k)/2} \leqslant C \lambda_{n,k}^{m+2}. \end{aligned}$$

Applying the last inequality, (4.6) and again the estimates for the ratio $\lambda_{n,i}/\lambda_{n,j}$ from Proposition 3.1, we obtain for $y \in I_{n,l}$

$$\begin{aligned} &\int_0^y \int_y^1 z(x)^{p-1} \, p_n(x,t)^p \, k_n(x,t)(t-y)^m \, dt\, dx \\ &\quad \leqslant \int_0^{t_{n,l}} \int_{t_{n,l-1}}^1 z(x)^{p-1} \, p_n(x,t)^p \, k_n(x,t)(t-t_{n,l-1})^m \, dt\, dx \\ &\quad \leqslant C^p \sum_{k=1}^{l} \sum_{j=l}^{n} \frac{\lambda_{n,k}^{1+(p-1)(m+2)} \lambda_{n,j} (\lambda_{n,j} + \cdots + \lambda_{n,l})^m}{(\lambda_{n,k} \vee \lambda_{n,j})^p} \, \theta^{(j-k)(p+1)/2} \\ &\quad \leqslant C^p \left(\sum_{k=1}^{l} \lambda_{n,k}^{(p-1)(m+1)} \, \theta^{(l-k)(p+1)/2} \right) \\ &\qquad \times \left(\lambda_{n,l}^{m+1} \sum_{j=l}^{n} (1+(j-l))^{\gamma+m(\gamma+1)} \, \theta^{(j-l)(p+1)/2} \right) \\ &\quad \leqslant C^p \lambda_{n,l}^{p(m+1)} \sum_{k=1}^{l} (1+(l-k))^{\gamma(p-1)(m+1)} \, \theta^{(l-k)(p+1)/2} \leqslant C^p \lambda_{n,l}^{p(m+1)}. \end{aligned}$$

This inequality and (4.2) give for $y \in I_{n,l}$ with $2 \leqslant l \leqslant n-1$

$$\int_0^y \int_y^1 z(x)^{p-1} \, p_n(x,t)^p \, k_n(x,t)(t-y)^m \, dt\, dx \leqslant C^p \frac{\varphi(y)^{p(m+1)}}{n^{p(m+1)}},$$

which, together with inequality (4.8), implies

$$\| T_{n,1}^{(m,\varphi)} f \|_p^p \leqslant \frac{C^p}{n^{p(m+1)}} \int_{t_{n,1}}^{t_{n,n-1}} \varphi(y)^{p(m+1)} \, |f^{(m+1)}(y)|^p \, dy.$$

The above estimates for $\|T_{n,i}^{(m,\varphi)}f\|_p$, $i=0, 1, 2, 3$, imply

$$\|T_n^{(m,\varphi)}f\|_p \leqslant \frac{C}{n^{m+1}} \|\varphi^{m+1} \cdot f^{(m+1)}\|_p,$$

with the constant C independent of p. Analogously we prove that

$$\|R_n^{(m,\varphi)}f\|_p \leqslant \frac{C}{n^{m+1}} \|\varphi^{m+1} \cdot f^{(m+1)}\|_p.$$

Thus, these inequalities and the decomposition (4.5) give

$$\|f - P_n^{(m,\varphi)}f\|_p \leqslant \frac{C}{n^{m+1}} \|\varphi^{m+1} \cdot f^{(m+1)}\|_p,$$

which completes the proof in case $0 \leqslant \beta(0), \beta(1) < 1$.

Let us discuss briefly the remaining cases. If $\beta(0), \beta(1) \geqslant 1$, then we have $(1/n)\,\varphi(x) \sim \lambda_{n,k}$ for all $k \in \Upsilon^*_{n,\varphi}$, $x \in I_{n,k}$, and the required estimate for $\|T_n^{(m,\varphi)}f\|_p$ can be obtained by the method similar to the one applied to $\|T_{n,1}^{(m,\varphi)}f\|_p$. If $\beta(0) \geqslant 1$ and $\beta(1) < 1$, then $(1/n)\,\varphi(x) \sim \lambda_{n,k}$ for $k \leqslant -1$, $x \in I_{n,k}$ and it is enough to split the integral defining $T_n^{(m,\varphi)}f(x)$ (over $[x, 1]$) into two parts: over $[x, 1] \cap [0, t_{n,-1}]$ and $[x, 1] \cap [t_{n,-1}, 1]$, and then treat the first part analogously to $T_{n,1}^{(m,\varphi)}f$, and the second part analogously to $T_{n,0}^{(m,\varphi)}f$. In case $\beta(0) < 1$ and $\beta(1) \geqslant 1$ we have $(1/n)\,\varphi(x) \sim \lambda_{n,k}$ for $k \geqslant 2$, $x \in I_{n,k}$, and then we split the integral defining $T_n^{(m,\varphi)}f(x)$ into two parts: The first part corresponding to integral over $[x, 1] \cap [t_{n,1}, 1]$ (which is treated analogously to $T_{n,1}^{(m,\varphi)}f$), and the second one corresponding to integral over $[x, 1] \cap [0, t_{n,1}]$. The second part is then further decomposed (using the property of reproducing of polynomials by $P_n^{(m,\varphi)}$) into parts analogous to $T_{n,2}^{(m,\varphi)}f$ and $T_{n,3}^{(m,\varphi)}f$.

Note that if $\beta(0) = 1$ or $\beta(1) = 1$, then n should be big enough to guarantee that for q_n from Proposition 3.1 and θ from Theorem 3.2 we have $q_n \cdot \theta < 1$. ∎

LEMMA 4.2. (Bernstein-Type Inequality). *Let φ be an admissible step-weight function and $m \in N$. Then there is a finite constant C such that for $n \geqslant 1$, $1 \leqslant l \leqslant m$, $1 \leqslant p \leqslant \infty$, and $f \in \mathcal{S}_{n,p}^{(m,\varphi)}$*

$$\|\varphi^l \cdot f^{(l)}\|_p \leqslant Cn^l \|f\|_p. \tag{4.9}$$

Moreover, there is a constant C such that for $n \geqslant 1$, $1 \leqslant p \leqslant \infty$, and $f \in \mathcal{S}_{n,p}^{(m,\varphi)}$

$$\omega_{\varphi,p}^{(m+1)}(f, \delta) \leqslant C \min(1, (n\delta)^{m+1/p}) \|f\|_p. \tag{4.10}$$

Proof. Let $f \in \mathcal{S}^{(m,\varphi)}_{n,p}$, $f = \sum_{i \in \Upsilon^{(m)}_{n,\varphi}} a_i N^{(m,\varphi)}_{n,i}$. Using formulae (3.10) for the derivatives of B-splines, L^p-stability of B-splines (cf. (3.9)), inequality (3.1) and the estimates of the ratio $\lambda_{n,k}/\lambda_{n,l}$ from Proposition 3.1, we get for $1 \leqslant l \leqslant m$ and for $p < \infty$

$$\begin{aligned}\|\varphi^l \cdot f^{(l)}\|_p &= \left(\sum_{k \in \Upsilon^*_{n,\varphi}} \int_{I_{n,k}} (\varphi(x)^l \cdot |f^{(l)}(x)|)^p \, dx \right)^{1/p} \\ &\leqslant C \left(\sum_{k \in \Upsilon^*_{n,\varphi}} n^{lp} \lambda^{lp+1}_{n,k} \sum_{i=k-m-2}^{k-1} \frac{|a_i|^p}{\lambda^{lp}_{n,k}} \right)^{1/p} \\ &\leqslant C n^l \left(\sum_{i \in \Upsilon^{(m)}_{n,\varphi}} \xi_{n,i} \, |a_i|^p \right)^{1/p} \\ &\leqslant C n^l \, \|f\|_p,\end{aligned}$$

with C independent of p. Passing with p to infinity completes the proof of (4.9).

To prove inequality (4.10), note that we can find constants A, a, depending on m and φ only, such that for $0 < h < a/n$

$$\operatorname{supp} \bar{\Delta}^{m+1}_{h\varphi} N^{(m,\varphi)}_{n,i} \subset [t_{n,i-1}, t_{n,i+m+2}], \qquad |\operatorname{supp} \bar{\Delta}^{m+1}_{h\varphi} N^{(m,\varphi)}_{n,i}| \leqslant Anh\xi_{n,i}.$$

Moreover, it can be checked that for $x \in \operatorname{supp} \bar{\Delta}^{m+1}_{h\varphi} N^{(m,\varphi)}_{n,i}$

$$|\bar{\Delta}^{m+1}_{h \cdot \varphi(x)} N^{(m,\varphi)}_{n,i}(x)| \leqslant C(nh)^m.$$

Now, the calculations similar to the ones from the first part of the proof give for $0 < h \leqslant a/n$

$$\|\bar{\Delta}^{m+1}_{h\varphi} f\|_p \leqslant C(nh)^{m+1/p} \left(\sum_{i \in \Upsilon^{(m)}_{n,\varphi}} |a_i|^p \, \xi_{n,i} \right)^{1/p} \leqslant C(nh)^{m+1/p} \, \|f\|_p,$$

which implies (4.10). ∎

THEOREM 4.3. *Let φ be an admissible step-weight function, $m \in N$ and $1 \leqslant p \leqslant \infty$. Then there are finite constant C and $n_0 \in N$ such that for $n \geqslant n_0$, $f \in L^p(I)$ ($f \in C(I)$ in case $p = \infty$)*

$$\|f - P^{(m,\varphi)}_n f\|_p \leqslant C\omega^{(m+1)}_{\varphi,p}\left(f, \frac{1}{n}\right).$$

Proof. Theorem 4.3 follows by standard arguments from Jackson type inequality (cf. Lemma 4.1) and the equivalence of the K-functional with the modulus of smoothness (cf. Theorem 2.1), with the help of the uniform (in n) bounds for the norms of the projection $P_n^{(m,\varphi)}$ (cf. Theorem 3.2). The details of the proof are omitted. ▮

For $1 \leqslant p \leqslant \infty$, $f \in L^p(I)$ ($f \in C(I)$ in case $p = \infty$) and $n \geqslant 1$, introduce the best approximation

$$E_{n,p}^{(m,\varphi)}(f) = \inf\{\|f - g\|_p : g \in \mathscr{S}_{n,p}^{(m,\varphi)}\}.$$

Recall that $P_n^{(m,\varphi)}$ is a projection on $\mathscr{S}_{n,p}^{(m,\varphi)}$, and by Theorem 3.2 there are M and n_0 such that $\|P_n^{(m,\varphi)}\|_p \leqslant M$ for $n \geqslant n_0$. Therefore, for $f \in L^p(I)$ ($f \in C(I)$ in case $p = \infty$) and $n \geqslant n_0$ we have

$$E_{n,p}^{(m,\varphi)}(f) \leqslant \|f - P_n^{(m,\varphi)} f\|_p \leqslant (M+1)\, E_{n,p}^{(m,\varphi)}(f). \tag{4.11}$$

THEOREM 4.4. *Let φ be an admissible step-weight function, $m \in N$ and $1 \leqslant p \leqslant \infty$. Then there are finite constant C and $\mu_0 \in N$ such that for $\mu \geqslant \mu_0$, $f \in L^p(I)$ ($f \in C(I)$ in case $p = \infty$)*

$$\omega_{\varphi,p}^{(l)}\left(f, \frac{1}{2^\mu}\right) \leqslant \frac{C}{2^{l\mu}}\left(\|f\|_p + \sum_{i=\mu_0}^{\mu} 2^{li} E_{2^i,p}^{(m,\varphi)}(f)\right) \qquad \text{for} \quad 1 \leqslant l \leqslant m,$$

$$\omega_{\varphi,p}^{(m+1)}\left(f, \frac{1}{2^\mu}\right) \leqslant \frac{C}{2^{(m+1/p)\mu}}\left(\|f\|_p + \sum_{i=\mu_0}^{\mu} 2^{(m+1/p)i} E_{2^i,p}^{(m,\varphi)}(f)\right).$$

Proof. Theorem 4.4 follows by standard arguments from Bernstein type inequalities (4.9) and (4.10) (cf. Lemma 4.2), and the details of the proof are omitted. ▮

As a consequence of Theorems 4.3 and 4.4 we get

THEOREM 4.5. *Let φ be an admissible step-weight function, $1 \leqslant p \leqslant \infty$, $m \in N$, $0 < \alpha < m$ and $f \in L^p(I)$ ($f \in C(I)$ in case $p = \infty$). Then the following conditions are equivalent.*

(i) $\omega_{\varphi,p}^{(m)}(f, \delta) = O(\delta^\alpha)$ *as* $\delta \to 0$;

(ii) $E_{n,p}^{(m,\varphi)}(f) = O(n^{-\alpha})$ *as* $n \to \infty$,

(iii) $\|f - P_n^{(m,\varphi)} f\|_p = O(n^{-\alpha})$ *as* $n \to \infty$.

Moreover, for $0 < \alpha < m + 1/p$, conditions (ii) *and* (iii) *are equivalent to*

(iv) $\omega_{\varphi,p}^{(m+1)}(f, \delta) = O(\delta^\alpha)$ *as* $\delta \to 0$.

These assertions remain valid when O is replaced by o.

4.2. *Order of Approximation by* $L_n^{(m,\varphi)}$ *and* $U_n^{(\varphi)}$

LEMMA 4.6 (Jackson-Type Inequality for $L_n^{(m,\varphi)}$). *Let* φ *be an admissible step-weight function and* $m \in N$. *Then there are finite constant* C *and* $n_0 \in N$ *such that for* $n \geqslant n_0$, $1 \leqslant p \leqslant \infty$, *and* $f \in W^1_{p,\varphi}(I)$

$$\|f - L_n^{(m,\varphi)} f\|_p \leqslant \frac{C}{n} \|\varphi \cdot f'\|_p.$$

Proof. The proof follows by the analogous ideas as the proof of Lemma 4.1 (with the use of the estimates for $\mathscr{L}_n^{(m,\varphi)}(s,t)$ from Proposition 3.3). ∎

Using Lemma 4.6 and the same arguments which imply Theorem 4.3 we obtain

THEOREM 4.7. *Let* φ *be an admissible step-weight function,* $m \in N$ *and* $1 \leqslant p \leqslant \infty$. *Then there are finite constant* C *and* $n_0 \in N$ *such that for* $n \geqslant n_0$ *and* $f \in L^p(I)$ ($f \in C(I)$ *in case* $p = \infty$)

$$\|f - L_n^{(m,\varphi)} f\|_p \leqslant C\omega^{(1)}_{\varphi,p}\left(f, \frac{1}{n}\right).$$

THEOREM 4.8. *Let* φ *be an admissible step-weight function. Then there are finite constant* C *and* $n_0 \in N$ *such that for* $n \geqslant n_0$ *and* $f \in C(I)$

$$\|f - U_n^{(\varphi)} f\|_\infty \leqslant C\omega^{(2)}_{\varphi,\infty}\left(f, \frac{1}{n}\right).$$

Proof. It follows from the uniform (in n) bounds for the norms $\|P_n^{(m,\varphi)}\|_\infty$ (cf. Theorem 3.2) and $\|U_n^{(\varphi)}\|_\infty = 1$ that

$$\|f - P_n^{(1,\varphi)} f\|_\infty \sim \|f - U_n^{(\varphi)} f\|_\infty \sim E^{(1,\varphi)}_{n,\infty}(f),$$

and Theorem 4.8 is a consequence of Theorem 4.3. ∎

As a consequence of Theorems 4.5, 4.7 and 4.8 we obtain

THEOREM 4.9. *Let* φ *be an admissible step-weight function,* $1 \leqslant p \leqslant \infty$, $m \in N$, $0 < \alpha < 1$ *and* $f \in L^p(I)$ ($f \in C(I)$ *in case* $p = \infty$). *Then the following conditions are equivalent*:

(i) $\omega^{(1)}_{\varphi, p}(f, \delta) = O(\delta^{\alpha})$ *as* $\delta \to 0$,

(ii) $\|f - L_n^{(m, \varphi)} f\|_p = O(n^{-\alpha})$ *as* $n \to \infty$.

In addition, for $p = \infty$ *the above conditions are equivalent to*

(iii) $\|f - U_n^{(\varphi)} f\|_{\infty} = O(n^{-\alpha})$ *as* $n \to \infty$.

The assertion is also valid when O *is replaced by* o.

Note that for $f \in C(I)$

$$U_n^{(\varphi)} f = \sum_{k \in \Upsilon^{(1)}_{n, \varphi}} f(t_{n, k+1})\, N^{(1, \varphi)}_{n, k}$$

and

$$U_{2n}^{(\varphi)} f - U_n^{(\varphi)} f = \sum_{k \in \Upsilon^{(1)}_{2n, \varphi} \cap 2Z} c_{n, k}(f)\, N^{(1, \varphi)}_{2n, k},$$

with

$$c_{n, k}(f) = -\lambda_{2n, k+1} \cdot \lambda_{2n, k+2} \cdot [t_{2n, k}, t_{2n, k+1}, t_{2n, k+2}; f].$$

Thus we get from Theorem 4.9:

COROLLARY 4.10. *Let* φ *be an admissible step-weight function and* $0 < \alpha < 1$. *Then for* $f \in C(I)$ *the following conditions are equivalent*:

(i) $\omega^{(1)}_{\varphi, \infty}(f, \delta) = O(\delta^{\alpha})$ *as* $\delta \to 0$,

(ii) $\sup_{n \in N} \sup_{k \in \Upsilon^{(1)}_{2n, \varphi} \cap 2Z} n^{\alpha} |c_{n, k}(f)| < \infty$.

5. SPLINES WITH TCHEBYSHEV KNOTS ON [0, 1]

In this section we consider spline spaces associated with one particular step-weight function, namely $\varphi(x) = \sqrt{x(1-x)}$. The modulus of smoothness $\omega^{(m)}_{\varphi, p}(f, t)$ gives the characterization of order of approximation by algebraic polynomials. Denote by Π_n the space of algebraic polynomials of degree $\leqslant n$, and for $f \in L^p(I)$ ($f \in C(I)$ in case $p = \infty$) introduce the best polynomial approximation

$$\mathscr{E}_{n, p}(f) = \inf\{\|f - g\|_p : g \in \Pi_n\}.$$

It is known (see [8, Theorems 7.2.1 and 7.2.4]) that for any $m \in N$ and $1 \leqslant p \leqslant \infty$ there is a constant C such that

$$\mathscr{E}_{n,p}(f) \leqslant C\omega_{\varphi,p}^{(m)}\left(f, \frac{1}{n}\right) \qquad \text{for} \quad n > m, \tag{5.1}$$

$$\omega_{\varphi,p}^{(m)}\left(f, \frac{1}{n}\right) \leqslant \frac{C}{n^m} \sum_{k=0}^{n} (k+1)^{m-1} \mathscr{E}_{k,p}(f). \tag{5.2}$$

The asymptotics of $\omega_{\varphi,p}^{(m)}(f, t)$ can be characterized by the order of approximation by spline functions with knots uniformly distributed with respect to the measure $dx/\varphi(x)$. For $\varphi(x) = \sqrt{x(1-x)}$ we have

$$\Upsilon_{n,\varphi} = \{0, ..., n\} \qquad \text{and} \qquad t_{n,k} = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{(n-k)\pi}{n}\right) \qquad \text{for} \quad 0 \leqslant k \leqslant n. \tag{5.3}$$

Note that

$$\lambda_{n,k} = \sin\left(\frac{\pi}{2n}\right)\sin\left(\frac{(2k-1)\pi}{2n}\right), \qquad 1 \leqslant k \leqslant n, \tag{5.4}$$

and

$$\frac{1}{3(1+|k-l|)} \leqslant \frac{\lambda_{n,k}}{\lambda_{n,l}} \leqslant 3(1+|k-l|) \qquad \text{for all} \quad n \geqslant 1, \quad 1 \leqslant k, \quad l \leqslant n.$$

As a consequence of inequalities (5.1) and (5.2), and Theorems 4.3 and 4.4 we get

Corollary 5.1. *Let* $\varphi(x) = \sqrt{x(1-x)}$, $m \in N$ *and* $1 \leqslant p \leqslant \infty$. *Then, there are* C *and* μ_0 *such that for* $\mu \geqslant \mu_0$ *and* $f \in L^p(I)$ ($f \in C(I)$ *in case* $p = \infty$) *the following inequalities hold*:

$$\mathscr{E}_{2^\mu,p}(f) \leqslant \frac{C}{2^{(m+1/p)\mu}}\left(\|f\|_p + \sum_{k=\mu_0}^{\mu} 2^{(m+1/p)k} E_{2^k,p}^{(m,\varphi)}(f)\right),$$

$$E_{2^\mu,p}^{(m,\varphi)}(f) \leqslant \frac{C}{2^{(m+1)\mu}}\left(\|f\|_p + \sum_{k=\mu_0}^{\mu} 2^{(m+1)k} \mathscr{E}_{2^k,p}(f)\right).$$

Consequently, for $0 < \alpha < m + 1/p$

$$\mathscr{E}_{n,p}(f) = O(n^{-\alpha}) \qquad \textit{iff} \quad E_{n,p}^{(m,\varphi)}(f) = O(n^{-\alpha})$$

and

$$\mathscr{E}_{n,p}(f) = o(n^{-\alpha}) \qquad \textit{iff} \quad E_{n,p}^{(m,\varphi)}(f) = o(n^{-\alpha}).$$

Finally, let us compare the order of approximation by positive spline operators $L_n^{(m,\varphi)} f$ and the order of approximation by some positive polynomial operators, namely Bernstein, Bernstein–Kantorovitch, and Bernstein–Durrmeyer operators. Bernstein, Bernstein–Kantorovitch, and Bernstein–Durrmeyer operators $B_n f$, $B_n^* f$, and $D_n f$ are defined by the respective formulae

$$B_n f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_{n,k}(x) \qquad \text{for} \quad f \in C(I),$$

$$B_n^* f(x) = \sum_{k=0}^{n} (n+1) \int_{k/(n+1)}^{(k+1)/(n+1)} f(u)\, du\, b_{n,k}(x) \qquad \text{for} \quad f \in L^p(I),$$

$$D_n f(x) = \sum_{k=0}^{n} (n+1) \int_0^1 f(u)\, b_{n,k}(u)\, du\, b_{n,k}(x) \qquad \text{for} \quad f \in L^p(I),$$

where

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

Let us mention some of the properties of the operators $B_n^* f$ and $D_n f$:

$$\|B_n^* f\|_p \leqslant \|f\|_p \qquad \text{and} \qquad \|D_n f\|_p \leqslant \|f\|_p \qquad \text{for} \quad f \in L^p(I), \quad 1 \leqslant p \leqslant \infty,$$

$$\int_0^1 B_n^* f(x)\, dx = \int_0^1 D_n f(x)\, dx = \int_0^1 f(x)\, dx \qquad \text{for} \quad f \in L^1(I).$$

As $B_n^* f$ and $D_n f$ are positive operators, this means that they take probability densities supported on I into probability densities. This property makes them useful for nonparametric density estimation (cf. for example [3, 4]).

It is known that for $0 < \alpha < 2$ (cf. [8, Chapter 9]; see also [10, 12–13])

$$\|f - B_n f\|_\infty = O(n^{-\alpha/2}) \qquad \text{iff} \quad \omega^{(2)}_{\varphi,\infty}(f, t) = O(t^\alpha), \tag{5.5}$$

$$\|f - B_n^* f\|_p = O(n^{-\alpha/2}) \qquad \text{iff} \quad \omega^{(2)}_{\varphi,p}(f, t) = O(t^\alpha), \tag{5.6}$$

and (cf. [7])

$$\|f - D_n f\|_p = O(n^{-\alpha/2}) \qquad \text{iff} \quad \omega^{(2)}_{\varphi,p}(f, t) = O(t^\alpha). \tag{5.7}$$

It follows from (5,5)–(5.7) and Theorem 4.9 that for $0 < \alpha < 1$ and $1 \leqslant p \leqslant \infty$ the conditions $\|f - B_n^* f\|_p = O(n^{-\alpha/2})$, $\|f - D_n f\|_p = O(n^{-\alpha/2})$ and $\|f - L_n^{(m,\varphi)} f\|_p = O(n^{-\alpha})$ are equivalent; in addition, for $p = \infty$, they are equivalent to $\|f - B_n f\|_\infty = O(n^{-\alpha/2})$. To obtain analogous equivalence for $1 \leqslant \alpha < 2$ we need more precise results on the order of approximation by operators $L_n^{(m,\varphi)} f$. We have the following

THEOREM 5.2. *Let $\varphi(x) = \sqrt{x(1-x)}$, $m \in N$ and $1 \leqslant p < \infty$. Then there is a constant C such that for $n \in N$ and $f \in W^2_{p,\varphi}(I)$*

$$\|f - L_n^{(m,\varphi)} f\|_p \leqslant \frac{C}{n^2} (\|f\|_p + \|\varphi^2 \cdot f''\|_p). \tag{5.8}$$

Consequently, there are constant C and $n_0 \in N$ such that for $f \in L^p(I)$ and $n \geqslant n_0$

$$\|f - L_n^{(m,\varphi)} f\|_p \leqslant C \left(\frac{\|f\|_p}{n^2} + \omega^{(2)}_{\varphi,p}\left(f, \frac{1}{n}\right)\right). \tag{5.9}$$

Remark. This result should be compared with the orders of approximation for the operators $D_n f$ and $B_n^* f$: Let $1 \leqslant p < \infty$; then there is a constant C such that for $f \in L^p(I)$ and $n \in N$

$$\|f - D_n f\|_p \leqslant C \left(\frac{\|f\|_p}{n} + \omega^{(2)}_{\varphi,p}\left(f, \frac{1}{\sqrt{n}}\right)\right),$$

$$\|f - B_n^* f\|_p \leqslant C \left(\frac{\|f\|_p}{n} + \omega^{(2)}_{\varphi,p}\left(f, \frac{1}{\sqrt{n}}\right)\right),$$

(cf. [7, Theorem 7.4] and [8, Theorem 9.3.2]).

Proof of Theorem 5.2. The idea of the proof is analogous to the argument used for Jackson type inequalities for $P_n^{(m,\varphi)}$ and $L_n^{(m,\varphi)}$ (Lemmas 4.1 and 4.6), but now we must take into account that operators $L_n^{(m,\varphi)}$ do not reproduce linear functions.

Let $1 \leqslant p < \infty$. By the density argument, it is enough to prove (5.8) for $f \in C^2(I)$. For $f \in C^2(I)$ we have

$$f(x) = f(0) \cdot (1-x) + f(1) \cdot x - \int_0^1 U(x, y)\, f''(y)\, dy,$$

where $U(x, y) = \min(x, y) \cdot \min(1-x, 1-y)$. Denote $Id(x) = x$ and $A_n^{(m)}(x) = Id(x) - L_n^{(m,\varphi)}(Id)(x)$. As the operator $L_n^{(m,\varphi)}$ reproduces constant functions, we have

$$\begin{aligned} L_n^{(m,\varphi)} f(x) &= f(0) + (f(1) - f(0))\, L_n^{(m,\varphi)}(Id)(x) \\ &\quad - \int_0^1 L_n^{(m,\varphi)}(U(\cdot, y))(x)\, f''(y)\, dy, \end{aligned}$$

and consequently

$$f(x)-L_n^{(m,\varphi)}f(x)=(f(1)-f(0))\,A_n^{(m)}(x)$$
$$-\int_0^1 f''(y)(U(x,y)-L_n^{(m,\varphi)}(U(\cdot,y))(x))\,dy. \quad (5.10)$$

Note for given y the function $U(\cdot,y)$ is linear on $[0,y]$ and $[y,1]$. Therefore, by the definition of $L_n^{(m,\varphi)}$ (cf. (3.17)) we have

$$L_n^{(m,\varphi)}(U(\cdot,y))(x)=(1-y)(x-A_n^{(m)}(x))+\int_y^1 (y-u)\,\mathscr{L}_n^{(m,\varphi)}(x,u)\,du$$
$$=y(1-x+A_n^{(m)}(x))+\int_0^y (u-y)\,\mathscr{L}_n^{(m,\varphi)}(x,u)\,du.$$

Putting into (5.10) the first of these formulae for $x\leqslant y\leqslant 1$ and the second one for $0\leqslant y<x$ we get

$$f(x)-L_n^{(m,\varphi)}f(x)=f'(x)\cdot A_n^{(m)}(x)-R_n^{(m)}f(x)-T_n^{(m)}f(x), \quad (5.11)$$

where

$$R_n^{(m)}f(x)=\int_0^x\int_0^y (y-u)\,\mathscr{L}_n^{(m,\varphi)}(x,u)\,du\,f''(y)\,dy,$$
$$T_n^{(m)}f(x)=\int_x^1\int_y^1 (u-y)\,\mathscr{L}_n^{(m,\varphi)}(x,u)\,du\,f''(y)\,dy.$$

Introduce two auxiliary operators, $\tilde{R}_n^{(m)}f$ and $\tilde{T}_n^{(m)}f$, defined as

$$\tilde{R}_n^{(m)}f(x)=0 \qquad \text{for} \quad 0\leqslant x<t_{n,n-1},$$

while for $t_{n,n-1}\leqslant x\leqslant 1$

$$\tilde{R}_n^{(m)}f(x)=\int_{t_{n,n-1}}^x (y-x)\,f''(y)\,dy$$
$$-\int_{t_{n,n-1}}^x\int_y^1 (y-u)\,\mathscr{L}_n^{(m,\varphi)}(x,u)\,du\,f''(y)\,dy,$$

and $\tilde{T}_n^{(m)}f(x)=0$ for $t_{n,1}\leqslant x\leqslant 1$, while for $0\leqslant x<t_{n,1}$

$$\tilde{T}_n^{(m)}f(x)=\int_x^{t_{n,1}} (x-y)\,f''(y)\,dy$$
$$-\int_x^{t_{n,1}}\int_0^y (u-y)\,\mathscr{L}_n^{(m,\varphi)}(x,u)\,du\,f''(y)\,dy.$$

Using again the fact that $L_n^{(m,\varphi)}$ reproduces constant functions and the definition of $A_n^{(m)}$, we get for $t_{n,n-1} \leqslant x \leqslant 1$

$$\int_{t_{n,n-1}}^{x} \int_0^y (y-u)\, \mathscr{L}_n^{(m,\varphi)}(x,u)\, du\, f''(y)\, dy$$
$$= A_n^{(m)}(x)(f'(x) - f'(t_{n,n-1})) + \tilde{R}_n^{(m)} f(x),$$

and for $0 \leqslant x \leqslant t_{n,1}$

$$\int_x^{t_{n,1}} \int_y^1 (u-y)\, \mathscr{L}_n^{(m,\varphi)}(x,u)\, du\, f''(y)\, dy$$
$$= A_n^{(m)}(x)(f'(x) - f'(t_{n,1})) + \tilde{T}_n^{(m)} f(x).$$

Now, define

$$\eta_n^{(m)} f(x) = \begin{cases} f'(t_{n,1}) & \text{for} \quad 0 \leqslant x \leqslant t_{n,1}, \\ f'(x) & \text{for} \quad t_{n,1} \leqslant x \leqslant t_{n,n-1}, \\ f'(t_{n,n-1}) & \text{for} \quad t_{n,n-1} \leqslant x \leqslant 1, \end{cases}$$

$$Q_n^{(m)} f(x) = \begin{cases} R_n^{(m)} f(x) + T_n^{(m)} f(t_{n,1}) + \tilde{T}_n^{(m)} f(x) & \text{for} \quad 0 \leqslant x \leqslant t_{n,1}, \\ R_n^{(m)} f(x) + T_n^{(m)} f(x) & \text{for} \quad t_{n,1} < x < t_{n,n-1}, \\ R_n^{(m)} f(t_{n,n-1}) + \tilde{R}_n^{(m)} f(x) + T_n^{(m)} f(x) & \text{for} \quad t_{n,n-1} \leqslant x \leqslant 1. \end{cases}$$

By the previous calculations and the definitions of $Q_n^{(m)} f$ and $\eta_n^{(m)} f(x)$, we can rewrite formula (5.11) as

$$f(x) - L_n^{(m,\varphi)} f(x) = \eta_n^{(m)} f(x) \cdot A_n^{(m)}(x) - Q_n^{(m)} f(x). \tag{5.12}$$

Applying the method used in the proof of Lemma 4.1, we check that there is a finite constant C, independent of $1 \leqslant p \leqslant \infty$, such that for $n \in N$ and $f \in W_{p,\varphi}^2(I)$

$$\|Q_n^{(m)} f\|_p \leqslant \frac{C}{n^2} \|\varphi^2 \cdot f''\|_p. \tag{5.13}$$

Therefore, it is sufficient to obtain the bound for $\|\eta_n^{(m)} f(\cdot)\, A_n^{(m)}(\cdot)\|_p$.

Applying the formulae (cf. [11, Chapter 4])

$$Id(x) = \sum_{i \in \Upsilon_{n,\varphi}^{(m)}} \frac{\sum_{j=1}^m t_{n,i+j}}{m} N_{n,i}^{(m,\varphi)}(x), \tag{5.14}$$

$$\int_0^1 Id(x)\, M_{n,i}^{(m,\varphi)}(x)\, dx = \frac{\sum_{j=0}^{m+1} t_{n,i+j}}{m+2} \tag{5.15}$$

(recall that $t_{n,i}=0$ for $i\leqslant 0$ and $t_{n,i}=1$ for $i\geqslant n$, cf. Section 3.2, while for $0\leqslant i\leqslant n$ the point $t_{n,i}$ is given by formula (5.3)), we get

$$L_n^{(m,\varphi)}(Id)(x)=\sum_{i\in\Upsilon_{n,\varphi}^{(m)}}\frac{\sum_{j=0}^{m+1}t_{n,i+j}}{m+2}\,N_{n,i}^{(m,\varphi)}(x),$$

and

$$A_n^{(m)}(x)=\sum_{i\in\Upsilon_{n,\varphi}^{(m)}}w_{n,i}^{(m)}\,N_{n,i}^{(m,\varphi)}(x),$$

where

$$\begin{aligned}w_{n,i}^{(m)}&=\frac{1}{m(m+2)}\sum_{j=1}^{m+1}(m-2j+2)\,\lambda_{n,i+j}\\&=\frac{1}{m(m+2)}\sum_{j=1}^{[(m+1)/2]}(m-2j+2)(\lambda_{n,i+j}-\lambda_{n,i+m+2-j}).\end{aligned}$$

It follows from formula (5.4) that $|\lambda_{n,i+j}-\lambda_{n,i+m+2-j}|=O(n^{-2})$, and moreover for i such that $|n/2-i|>n/4$ the differences $\lambda_{n,i+j}-\lambda_{n,i+m+2-j}$, $j=1,\ldots,[(m+1)/2]$, have the same sign and $|\lambda_{n,i+j}-\lambda_{n,i+m+2-j}|\sim n^{-2}$. Therefore, the above formula for $w_{n,i}^{(m)}$ gives

$$|w_{n,i}^{(m)}|=O(n^{-2})\quad\text{for } i\in\Upsilon_{n,\varphi}^{(m)}\quad\text{and}\quad|w_{n,i}^{(m)}|\sim n^{-2}\quad\text{for } |n/2-i|\geqslant n/4,$$

whence (cf. (3.9))

$$\|A_n^{(m)}\|_p\sim n^{-2},\qquad 1\leqslant p\leqslant\infty. \tag{5.16}$$

Now, we obtain the bound for $(\int_0^{1/2}|\eta_n^{(m)}f(x)\,A_n^{(m)}(x)|^p\,dx)^{1/p}$; the other integral $(\int_{1/2}^1|\eta_n^{(m)}f(x)\,A_n^{(m)}(x)|^p\,dx)^{1/p}$ can be treated analogously. Denote

$$a(f)=4\int_{1/4}^{1/2}(y-\tfrac{1}{4})\,f''(y)\,dy,\qquad b(f)=4\int_{1/4}^{1/2}f'(y)\,dy.$$

Integrating by parts we get $a(f)=f'(\tfrac{1}{2})-b(f)$; moreover, observe that

$$|a(f)|\leqslant C\,\|\varphi^2\cdot f''\|_1\leqslant C\,\|\varphi^2\cdot f''\|_p$$

and

$$\begin{aligned}|b(f)| &\leqslant 4 \int_{1/4}^{1/2} |f'(y)|\, dy \\ &\leqslant C\left(\int_{1/4}^{1/2} |f(y)|\, dy + \int_{1/4}^{1/2} |f''(y)|\, dy\right) \\ &\leqslant C(\|f\|_1 + \|\varphi^2 \cdot f''\|_1) \leqslant C(\|f\|_p + \|\varphi^2 \cdot f''\|_p).\end{aligned}$$

Now we have

$$\eta_n^{(m)} f(x) = (\eta_n^{(m)} f(x) - f'(\tfrac{1}{2})) + a(f) + b(f),$$

which, together with (5.16) and the above estimates for $|a(f)|$ and $|b(f)|$, implies

$$\begin{aligned}&\left(\int_0^{1/2} |\eta_n^{(m)} f(x)\, A_n^{(m)}(x)|^p\, dx\right)^{1/p} \\ &\quad\leqslant \frac{C}{n^2}(\|f\|_p + \|\varphi^2 \cdot f''\|_p) + \left(\int_0^{1/2} \left|\left(\eta_n^{(m)} f(x) - f'\left(\frac{1}{2}\right)\right) A_n^{(m)}(x)\right|^p dx\right)^{1/p}\end{aligned} \tag{5.17}$$

Denote $q = p/(p-1)$; now we have for $0 \leqslant x \leqslant t_{n,1}$

$$\begin{aligned}|\eta_n^{(m)} f(x) - f'(\tfrac{1}{2})| = |f'(t_{n,1}) - f'(\tfrac{1}{2})| &\leqslant \int_{t_{n,1}}^{1/2} |f''(y)|\, dy \\ &\leqslant \left(\int_{t_{n,1}}^{1/2} y^p\, |f''(y)|^p\, dy\right)^{1/p} \left(\int_{t_{n,1}}^{1/2} y^{-q}\right)^{1/q} \\ &\leqslant C n^{2/p}\, \|\varphi^2 \cdot f''\|_p.\end{aligned}$$

(It should be noted that the constant C in the last sequence of inequalities depends on p.) As $t_{n,1} = O(n^{-2})$ and $|A_n^{(m)}(x)| = O(n^{-2})$, the last inequality implies that

$$\left(\int_0^{t_{n,1}} \left|\left(\eta_n^{(m)} f(x) - f'\left(\frac{1}{2}\right)\right) A_n^{(m)}(x)\right|^p dx\right)^{1/p} \leqslant \frac{C}{n^2}\, \|\varphi^2 \cdot f''\|_p. \tag{5.18}$$

To estimate the integral over $[t_{n,1}, \frac{1}{2}]$, let $n' = [(n+1)/2]$. It follows from formula (5.4) that for $1 \leqslant k \leqslant n'$ we have $\lambda_{n,k} \sim k/n^2$. Therefore, using inequality (3.2) from Proposition 3.1 and applying twice Jensen's inequality (at first, to $(\sum_{j=i}^{n'} \cdots)^p$, with weights $i^{1/p}/j^{1+1/p}$, and then to the integral over $[t_{n,j-1}, t_{n,j}]$, with the measure $n\, dy/\varphi(y)$) we get

$$\int_{t_{n,1}}^{1/2} \left| \left(\eta_n^{(m)} f(x) - f'\left(\frac{1}{2}\right)\right)\right|^p dx$$

$$\leqslant \int_{t_{n,1}}^{t_{n,n'}} \left(\int_x^{t_{n,n'}} |f''(y)|\, dy \right)^p dx$$

$$\leqslant \sum_{i=2}^{n'} \lambda_{n,i} \left(\sum_{j=i}^{n'} \int_{t_{n,j-1}}^{t_{n,j}} |f''(y)|\, dy \right)^p$$

$$\leqslant C \sum_{i=2}^{n'} \lambda_{n,i} \left(\sum_{j=i}^{n'} \int_{t_{n,j-1}}^{t_{n,j}} \left(\frac{\varphi(y)}{n\lambda_{n,j}} \right)^{1+1/p} |f''(y)|\, dy \right)^p$$

$$\leqslant Cn^{p-1} \sum_{i=2}^{n'} i \left(\sum_{j=i}^{n'} \frac{1}{j^{1+1/p}} \int_{t_{n,j-1}}^{t_{n,j}} \varphi(y)^{1+1/p}\, |f''(y)|\, dy \right)^p$$

$$\leqslant Cn^{p-1} \sum_{i=2}^{n'} \sum_{j=i}^{n'} \frac{i^{1/p}}{j^{1+1/p}} \left(\int_{t_{n,j-1}}^{t_{n,j}} \varphi(y)^{2+1/p}\, |f''(y)|\, \frac{dy}{\varphi(y)} \right)^p$$

$$\leqslant C \sum_{i=2}^{n'} \sum_{j=i}^{n'} \frac{i^{1/p}}{j^{1+1/p}} \int_{t_{n,j-1}}^{t_{n,j}} \varphi(y)^{2p}\, |f''(y)|^p\, dy$$

$$\leqslant C \int_{t_{n,1}}^{t_{n,n'}} \varphi(y)^{2p}\, |f''(y)|^p\, dy.$$

(The constant C in the above sequence of inequalities depends again on p.) As $|A_n^{(m)}(x)| = O(n^{-2})$, the above calculations imply

$$\left(\int_{t_{n,1}}^{1/2} \left| \left(\eta_n^{(m)} f(x) - f'\left(\frac{1}{2}\right)\right) A_n^{(m)}(x) \right|^p dx \right)^{1/p} \leqslant \frac{C}{n^2} \|\varphi^2 \cdot f''\|_p,$$

which, together with (5.17) and (5.18), gives

$$\left(\int_0^{1/2} |\eta_n^{(m)} f(x)\, A_n^{(m)}(x)|^p\, dx \right)^{1/p} \leqslant \frac{C}{n^2} (\|f\|_p + \|\varphi^2 \cdot f''\|_p).$$

The integral over $[\frac{1}{2}, 1]$ is treated analogously, so we get

$$\|\eta_n^{(m)} f \cdot A^{(m)}\|_p \leqslant \frac{C}{n^2} (\|f\|_p + \|\varphi^2 \cdot f''\|_p).$$

This inequality, (5.12) and (5.13) imply that for $1 \leqslant p < \infty$ and $f \in C^2(I)$

$$\|f - L_n^{(m,\varphi)} f\|_p \leqslant \frac{C}{n^2} (\|f\|_p + \|\varphi^2 \cdot f''\|_p),$$

where the constant C depends only on p. The density argument gives (5.8) for all $f \in W^2_{p,\varphi}(I)$.

Finally, inequality (5.8) and the standard argument with the equivalence of K-functional and modulus of smoothness (i.e., Theorem 2.1) imply (5.9). ∎

Remark. Repeating the above calculations for $p=\infty$ we obtain that there is a constant C such that for $f \in W^2_{\infty,\varphi}(I)$

$$\|f - L_n^{(m,\varphi)} f\|_\infty \leqslant C\left(\frac{\|f\|_p}{n^2} + \frac{\log n}{n^2}\,\|\varphi^2 \cdot f''\|_\infty\right), \tag{5.19}$$

and consequently there is a constant $C>0$ such that for $f \in C(I)$

$$\|f - L_n^{(m,\varphi)} f\|_\infty \leqslant C\left(\frac{\|f\|_\infty}{n^2} + \log n\,\omega^{(2)}_{\varphi,\infty}\left(f, \frac{1}{n}\right)\right). \tag{5.20}$$

Moreover, the factor $\log n$ appearing on the right-hand side of inequalities (5.19) and (5.20) cannot be replaced by 1. To see this, consider the function $f(x) = x - x\log x$; then $f \in C(I)$, $f''(x) = -1/x$, which implies $\omega^{(2)}_{\varphi,\infty}(f, 1/n) = O(n^{-2})$. On the other hand, in the notation from the proof of Theorem 5.2,

$$\|Q_n f\|_\infty = O(n^{-2}), \qquad \text{while} \quad \|\eta_n^{(m)} f A_n^{(m)}\|_\infty \sim \frac{\log n}{n^2},$$

which gives

$$\|f - L_n^{(m,\varphi)} f\|_\infty \sim \frac{\log n}{n^2}.$$

Proposition 5.3. *Let* $m \in N$, $1 \leqslant p \leqslant \infty$, $\varphi(x) = \sqrt{x(1-x)}$. *Then for* $f \in L^p(I)$ ($f \in C(I)$ *in case* $p=\infty$) *and* $0 < \alpha < \min(m + 1/p, 2)$

$$\|f - L_n^{(m,\varphi)} f\|_p = O(n^{-\alpha}) \qquad \text{iff} \quad \omega^{(2)}_{\varphi,p}(f,\delta) = O(\delta^\alpha).$$

Proof. For $p<\infty$ the result follows from Theorems 4.4 and 5.2. To obtain the result for $p=\infty$ note that for $x \in (t_{n,k}, t_{n,k+1})$

$$\int_0^1 (t-x)^2\, \mathscr{L}_n^{(m,\varphi)}(x,t)\,dt \leqslant C \sum_{i=k-m}^{k} \sum_{j=0}^{m+1} (x - t_{n,i+j})^2$$
$$\leqslant \frac{C}{n^2}\left(\varphi(x)^2 + \frac{1}{n^2}\right);$$

in the above calculations we use (5.15) and the formula (cf. [11, Chapter 4])

$$\int_0^1 t^2 M_{n,i}^{(m,\varphi)}(t)\,dt = \frac{2}{(m+2)(m+3)} \sum_{0 \leqslant j \leqslant l \leqslant m+1} t_{n,i+j} t_{n,i+l}.$$

Note that $|x - L_n^{(m,\varphi)}(Id)(x)| = O(n^{-2})$ (cf. (5.16)); now, applying Theorem 4.4 and Theorem 5.1 of [6] we get the result for $p = \infty$. ∎

As a consequence of Proposition 5.3 and (5.5)–(5.7) we get

COROLLARY 5.4. *Let* $m \in N$, $1 \leqslant p \leqslant \infty$, $\varphi(x) = \sqrt{x(1-x)}$, $f \in L^p(I)$ ($f \in C(I)$ *in case* $p = \infty$) *and* $0 < \alpha < \min(m + 1/p, 2)$. *Then the following conditions are equivalent*:

(i) $\|f - L_n^{(m,\varphi)} f\|_p = O(n^{-\alpha})$,

(ii) $\|f - B_n^* f\|_p = O(n^{-\alpha/2})$,

(iii) $\|f - D_n f\|_p = O(n^{-\alpha/2})$.

For $p = \infty$ *the above conditions are also equivalent to*

(iv) $\|f - B_n f\|_p = O(n^{-\alpha/2})$.

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